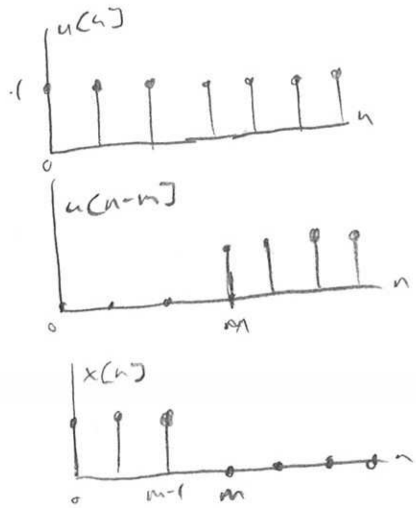
1.) (5.2-1)

$$x[n] = u[n] - u[n-m]$$



Definition of Z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$= \sum_{n=0}^{m-1} z^{-n} \quad \leftarrow \text{geometric series}$$

$$= \frac{z^{-((m-1)+1)} - z^{-0}}{z^{-1} - 1}$$

$$= \frac{z^{-m} - 1}{z^{-1} - 1} = \boxed{\frac{1 - z^{-m}}{1 - z^{-1}}}$$

Table and rules:

$$\mathcal{Z}\{u[n]\} = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}$$

$$\mathcal{Z}\{\phi[n-m]\, u[n-m]\} = z^{-m}\, \Phi(z)$$

pick $\phi[n] = u[n] \rightarrow \phi[n-m]\, u[n-m] = u^2[n-m] = u[n-m]$

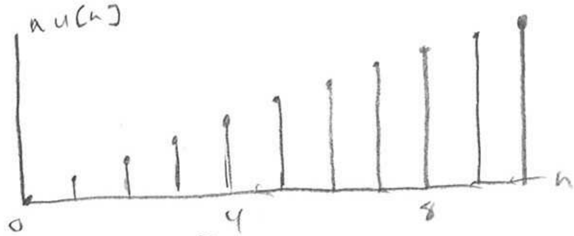
$$\mathcal{Z}\{u[n-m]\} = z^{-m} \frac{1}{1 - z^{-1}}$$

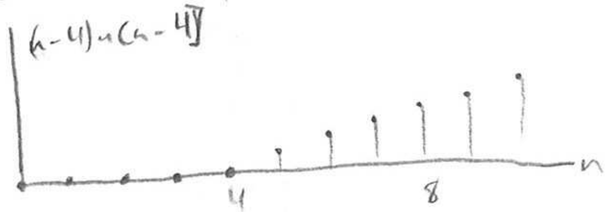
Z-transform is linear: $\mathcal{Z}\{u[n] - u[n-m]\} = \frac{1}{1 - z^{-1}} - z^{-m} \frac{1}{1 - z^{-1}} = \boxed{\frac{1 - z^{-m}}{1 - z^{-1}}}$

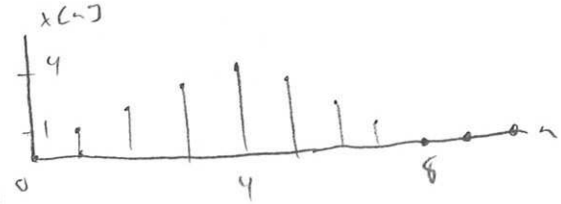
2) (5.2-2)

$x[n] = n\,u[n] - 2(n-4)\,u[n-4] + (n-8)\,u[n-8]$

$$= \begin{cases} n & 0 \le n \le 4 \\ 8-n & 5 \le n \le 8 \\ 0 & \text{else} \end{cases}$$







Definition:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} + 3z^{-5} + 2z^{-6} + z^{-7}$$

$$= \boxed{\left(z^6 + 2z^5 + 3z^4 + 4z^3 + 3z^2 + 2z + 1\right) z^{-7}}$$

Rules:

$$Z\{n\,u[n]\} = \frac{z}{(z-1)^2}$$

$$Z\{(n-m)\,u[n-m]\} = z^{-m} \frac{z}{(z-1)^2}$$

$$Z\{x[n]\} = \frac{z}{(z-1)^2} + -2z^{-4}\frac{z}{(z-1)^2} + z^{-8}\frac{z}{(z-1)^2}$$

$$= \boxed{\frac{z}{(z-1)^2}\left(1 - 2z^{-4} + z^{-8}\right)} = \frac{1}{z^7}\frac{1}{(z-1)^2}\left(\overbrace{z^8 - 2z^4 + 1}^{\text{quadratic in } z^4}\right)$$

$$z^8 - 2z^4 + 1 = \varrho^2 - 2\varrho + 1 = (\varrho - 1)^2 = (z^4 - 1)^2 \qquad (\varrho = z^4)$$

$$z^4 - 1 = (z^2 - 1)(z^2 + 1) = (z-1)(z+1)(z^2+1)$$

Observation: $z^4 - 1$ has roots at $\pm 1, \pm j$

$z^4 = 1 \rightarrow z = 1^{1/4}$
$= (e^{2\pi j n})^{1/4}$
$= e^{\pi j n/2}$

$$\left((z+1)(z^2+1)\right)^2 = (z^3 + z^2 + z + 1)^2 =$$

$$= (z^6 + z^5 + z^4 + z^3) + (z^5 + z^4 + z^3 + z^2) + (z^4 + z^3 + z^2 + z)$$

$$+ (z^3 + z^2 + z + 1)$$

$$= z^6 + 2z^5 + 3z^4 + 4z^3 + 3z^2 + 2z + 1$$

$$X(z) = z^{-7}\frac{1}{(z-1)^2}(z-1)^2(z+1)^2(z^2+1)^2 = z^{-7}\left(z^6 + 2z^5 + 3z^4 + 4z^3 + 3z^2 + 2z + 1\right) \ \boxed{\text{O.K.}}$$

3.a (5.2-3)

$$\gamma^n u[n] \xrightarrow{Z} \frac{z}{z-\gamma}$$

$$n x[n] \xrightarrow{Z} -z \frac{d}{dz} X(z)$$

$$n^2 x[n] \xrightarrow{Z} -z \frac{d}{dz}\left(-z \frac{d}{dz} X(z)\right)$$

$$= -z\left(-\frac{d}{dz} X(z) - z \frac{d^2}{dz^2} X(z)\right)$$

$$= \left(z^2 \frac{d^2}{dz^2} + z\frac{d}{dz}\right) X(z)$$

$$n^2 u[n] \xrightarrow{Z} z^2 \frac{d^2}{dz^2} \frac{z}{z-\gamma} + z \frac{d}{dz} \frac{z}{z-\gamma}$$

$$= z^2 \frac{d}{dz} \frac{(z-\gamma)-(z)}{(z-\gamma)^2} + z \frac{(z-\gamma)-z}{(z-\gamma)^2}$$

$$= z^2 \frac{0\cdot(z-\gamma)^2 - (-\gamma)2(z-\gamma)}{(z-\gamma)^4} + \frac{-\gamma z}{(z-\gamma)^2}$$

$$= \frac{2\gamma z^2}{(z-\gamma)^3} + \frac{-\gamma z(z-\gamma)}{(z-\gamma)^3} = \frac{2\gamma z^2 - \gamma z^2 + \gamma^2 z}{(z-\gamma)^3}$$

$$= \frac{\gamma z(z+\gamma)}{(z-\gamma)^3}$$

set $\gamma = 1 \rightarrow \boxed{\frac{z(z+1)}{(z-1)^3}}$

3.b

I'll do it slightly differently. Obviously we could use the above.

$$\gamma^n u[n] \xrightarrow{Z} \frac{z}{z-\gamma}$$

$$n\gamma^n u[n] \xrightarrow{Z} -z\frac{d}{dz}\frac{z}{z-\gamma} = -z \frac{(z-\gamma)-(z)}{(z-\gamma)^2} = \frac{z\gamma}{(z-\gamma)^2}$$

$$\boxed{\frac{z\gamma(z+\gamma)}{(z-\gamma)^3}}$$

$$n^2\gamma^n u[n] = n(n\gamma^n u[n]) \xrightarrow{Z} -z\frac{d}{dz}\frac{z\gamma}{(z-\gamma)^2} = -z \frac{\gamma(z-\gamma)^2 - z\gamma 2(z-\gamma)}{(z-\gamma)^4} = \frac{-z\gamma(z-\gamma-2z)}{(z-\gamma)^3} =$$

3.c

$$n^3 u[n] \xrightarrow{Z} -z\frac{d}{dz}\frac{z(z+1)}{(z-1)^3} = -z\frac{(2z+1)(z-1)^3 - (z^2+z)3(z-1)^2}{(z-1)^6}$$

$$= -z\frac{(2z+1)(z-1) - (z^2+z)3}{(z-1)^4}$$

$$= -z\frac{2z^2 - z - 1 - 3z^2 - 3z}{(z-1)^4} = \boxed{\frac{z(z^2+4z+1)}{(z-1)^4}}$$

3.d

$$a^n u[n] \xrightarrow{Z} \frac{z}{z-a}$$

$$a^n u[n-m] = a^{n-m}\underbrace{a^m}_{\text{constant}} u[n-m] \xrightarrow{Z} a^m z^{-m}\frac{z}{z-a}$$

$$a^n(u[n] - u[n-m]) \xrightarrow{Z} \boxed{\frac{z}{z-a}\left(1 - \left(\frac{a}{z}\right)^m\right)}$$

3.e

$$n e^{-2n} u[n-m] = (n-m+m)e^{-2(n-m+m)} u[n-m]$$

$$= (n-m)(e^{-2})^{(n-m)} e^{-2m} u[n-m] + m e^{-2m}(e^{-2})^{(n-m)} u[n-m]$$

$$\xrightarrow{Z} \boxed{z^{-m}\frac{e^{-2}z}{(z-e^{-2})^2} e^{-2m} + m e^{-2m}\frac{z}{(z-e^{-2})} z^{-m}}$$

3.f

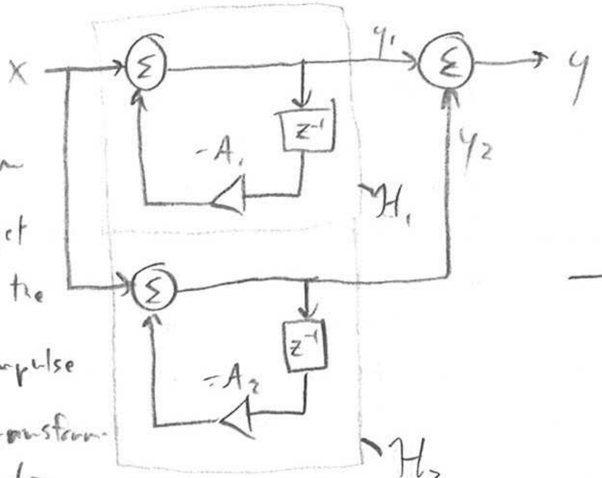
$$(n-2)\,2^{-(n-3)} u[n-4] = (n-4+2)\,2^{-(n-4+1)} u[n-4]$$

$$= 2^{-1}(n-4)\,2^{-(n-4)} u[n-4] + 2\cdot 2^{-1}\,2^{-(n-4)} u[n-4]$$

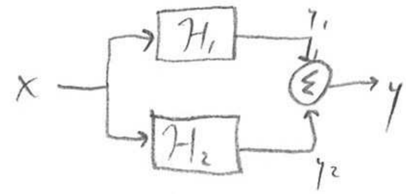
$$= \boxed{\frac{1}{2} z^{-4}\frac{2^{-1}z}{(z-2^{-1})^2} + z^{-4}\frac{z}{(z-\gamma)}}$$

4.a) (5.4-1)

We know we can get a difference equation from the diagram, and easily get a transfer function from the equation, and get the impulse response by inverse Z-transforming the transfer function.



$\longrightarrow$



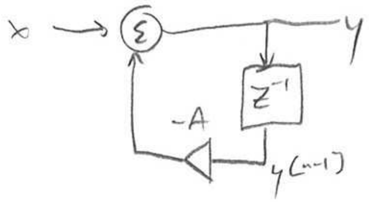
First recognize that the total system (call it $\mathcal{H}$) is the linear combination of 2 simpler systems ($\mathcal{H}_1$ and $\mathcal{H}_2$), both of which are identical in form.

Clearly the total output is the sum of outputs of the subsystems:

$$y = y_1 + y_2 = h_1 * x + h_2 * x = (h_1 + h_2) * x \xrightarrow{Z} (H_1 + H_2) \cdot X(z) = H(z)\,X(z)$$

so $H(z) = H_1(z) + H_2(z)$ and $h[n] = h_1[n] + h_2[n]$

Solve the subsystem:



$$\rightarrow \quad y[n] = x[n] + -A\,y[n-1] \Leftrightarrow y[n+1] + A\,y[n] = x[n+1]$$

$$\rightarrow Z \rightarrow \quad z\,Y(z) + A\,Y(z) = z\,X(z) \rightarrow \frac{Y}{X} = \frac{z}{z+A} = H_i(z)$$

$$H_{1,2}(z) = \frac{z}{z + A_{1/2}} \longrightarrow H(z) = \frac{z}{z+A_1} + \frac{z}{z+A_2} \rightarrow h[n] = (-A_1)^n u[n] + (-A_2)^n u[n]$$

$$\boxed{A_1 = \frac{-1 \pm j}{\sqrt{8}}, \quad A_2 = \frac{-1 \mp j}{\sqrt{8}}}$$

4.b

We will solve the system for $u[n]$, because it's easier. The system is LTI, so we can just shift our input and the output is shifted by the same amount. There are no initial conditions to worry about, so we will just convolve the input with the impulse response. (Or do it in the z-domain because it's easier).

$$u[n] \xrightarrow{z} \frac{z}{z-1} = X'(z) \qquad x'[n] = x[n-3]$$

$$h[n] \xrightarrow{z} H(z) = \frac{z}{z+A_1} + \frac{z}{z+A_2}$$

$$Y'(z) = H(z)X(z) = \underbrace{\frac{z^2}{(z-1)(z+A_1)}}_{Y_1'(z)} + \underbrace{\frac{z^2}{(z-1)(z+A_2)}}_{Y_2'(z)} \qquad y'[n] = y[n-3]$$

$$\frac{Y_i'(z)}{z} = \frac{z}{(z+A_i)(z-1)} = \frac{\alpha}{z+A_i} + \frac{\beta}{z-1}$$

$$\alpha = \lim_{z \to -A_i} \left(\frac{Y_i'(z)}{z}\right)(z+A_i) = \lim_{z \to -A_i} \frac{z}{z-1} = \frac{-A_i}{-A_i-1} = \frac{A_i}{A_i+1}$$

$$\beta = \lim_{z \to 1} \left(\frac{Y_i'(z)}{z}\right)(z-1) = \lim_{z \to 1} \frac{z}{z+A_i} = \frac{1}{1+A_i}$$

$$Y_i'(z) = \frac{z\alpha}{z+A_i} + \frac{z\beta}{z-1} = \frac{1}{1+A_i}\left(\frac{A_i z}{z+A_i} + \frac{z}{z-1}\right)$$

$$y_i'[n] = \frac{1}{1+A_i}\left(A_i(-A_i)^n + 1\right)u[n]$$

$$y'[n] = y_1'[n] + y_2'[n] = \frac{1}{1+A_1}\left(A_1(-A_1)^n + 1\right)u[n] + \frac{1}{1+A_2}\left(A_2(-A_2)^n + 1\right)u[n]$$

$$y[n] = y'[n+3] = \boxed{\frac{1}{1+A_1}\left(A_1(-A_n)^{n+3} + 1\right)u[n+3] + \frac{1}{1+A_2}\left(A_2(-A_2)^{n+3} + 1\right)u[n+3]}$$

5.a) (5.5-5)

$$y[n+1] - \tfrac{1}{2} y[n] = x[n+1] + 0.8\, x[n]$$

$$\xrightarrow{\mathcal{Z}} \quad (z - 0.5) Y(z) = (z + 0.8) X(z) \rightarrow \frac{Y(z)}{X(z)} = H(z) = \frac{z + 0.8}{z - 0.5}$$

$$H(e^{j\Omega}) = \frac{e^{j\Omega} + 0.8}{e^{j\Omega} - 0.5}$$

Amplitude response $= |H(e^{j\Omega})|$

$$|H(e^{j\Omega})|^2 = H(e^{j\Omega}) H^*(e^{j\Omega}) = \frac{e^{j\Omega} + 0.8}{e^{j\Omega} - 0.5} \; \frac{\overset{z^*}{e^{-j\Omega}} + 0.8}{e^{-j\Omega} - 0.5} =$$

$$= \frac{1 + 0.8(e^{j\Omega} + e^{-j\Omega}) + 0.8^2}{1 - 0.5(e^{j\Omega} + e^{-j\Omega}) + 0.5^2} = \frac{1 + 2 \cdot 0.8 \cos(\Omega) + 0.8^2}{1 - 0.5 \cdot 2\cos(\Omega) + 0.5^2}$$

$$= \frac{1.64 + 1.6 \cos(\Omega)}{1.25 - \cos(\Omega)}$$ ← clearly, this is real and non-negative

$$|H(e^{j\Omega})| = \sqrt{\frac{1.64 + 1.6\cos(\Omega)}{1.25 - \cos(\Omega)}}$$

phase response $= \arg(H(e^{j\Omega})) = \tan^{-1}\left(\frac{\operatorname{Im}(H(e^{j\Omega}))}{\operatorname{Re}(H(e^{j\Omega}))}\right)$

$$H(e^{j\Omega}) = \frac{e^{j\Omega} + 0.8}{e^{j\Omega} + 0.5} = \frac{e^{j\Omega} + 0.8}{D(\Omega)} = \frac{e^{j\Omega} + 0.8}{D(\Omega)} \; \frac{D^*(\Omega)}{D^*(\Omega)}$$

$$= \frac{(e^{j\Omega} + 0.8)(e^{-j\Omega} - 0.5)}{|D(\Omega)|^2} = \frac{1 + 0.8e^{-j\Omega} - 0.5e^{j\Omega} - 0.4}{|D(\Omega)|^2}$$

$e^{\pm j\Omega} = \cos\Omega \pm j \sin\Omega$

$$= \frac{0.6 + (0.8 - 0.5)\cos(\Omega) + j(-0.5 - 0.8)\sin(\Omega)}{|D(\Omega)|^2}$$

$$= \frac{1}{|D(\Omega)|^2}\left(0.6 + 0.3\cos(\Omega) + j(-1.3)\sin(\Omega)\right)$$

$$\arg(H(e^{j\Omega})) = \tan^{-1}\left(\frac{-1.3\sin(\Omega)}{0.6 + 0.3\cos(\Omega)}\right)$$

I really should have said "and plot the responses."

5.b

$x = \cos(0.5n - \pi/3)$ is the linear combination of eigen signals.

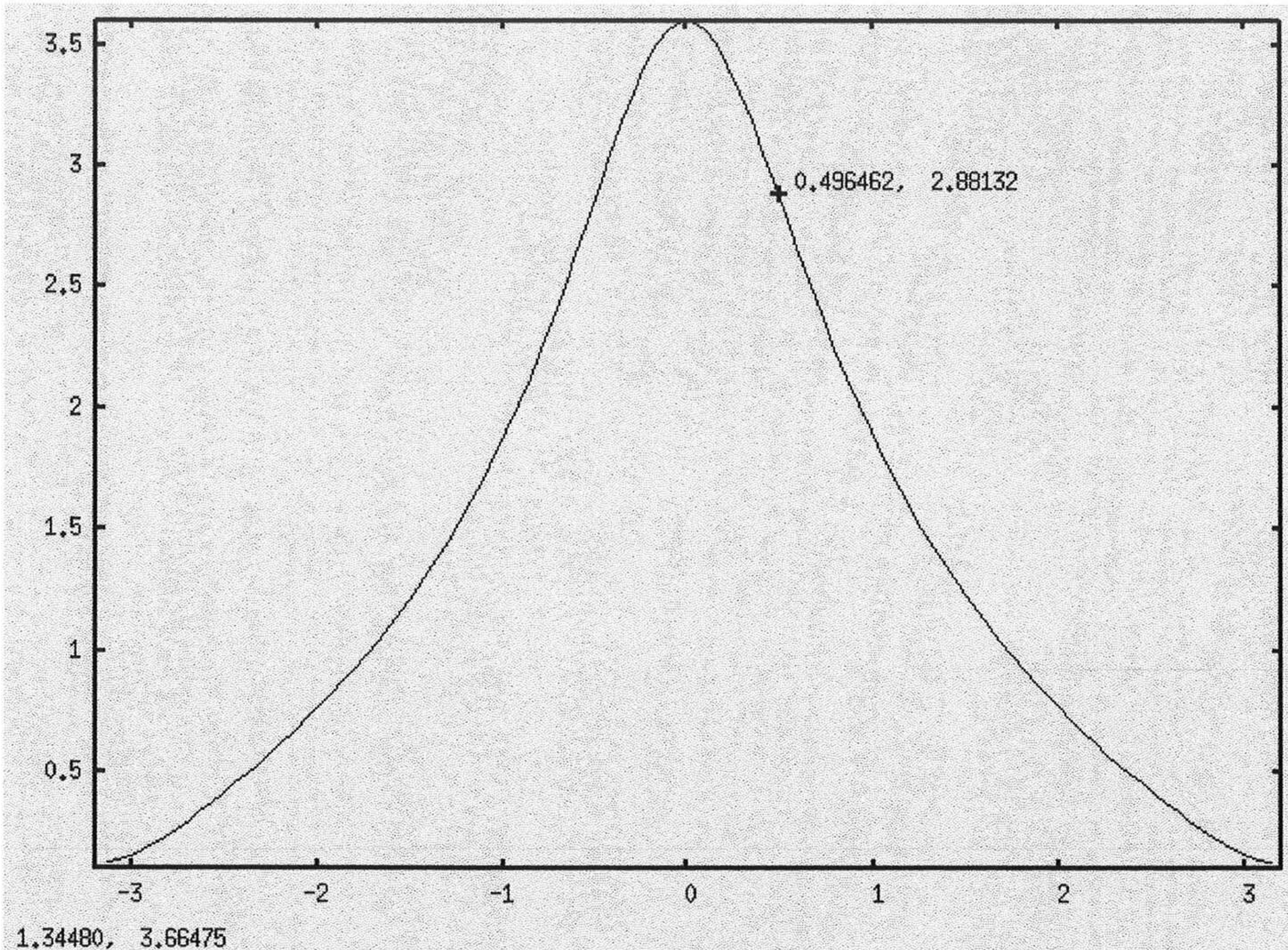
$$x[n] = \frac{e^{j(0.5n - \pi/3)} + e^{-j(0.5n - \pi/3)}}{2} = \underbrace{\frac{1}{2}e^{-j\pi/3}}_{\text{constant}}\underbrace{\left(e^{j1/2}\right)^n}_{\text{eigen signal}} + \frac{1}{2}e^{+j\pi/3}\left(e^{-j1/2}\right)^n$$

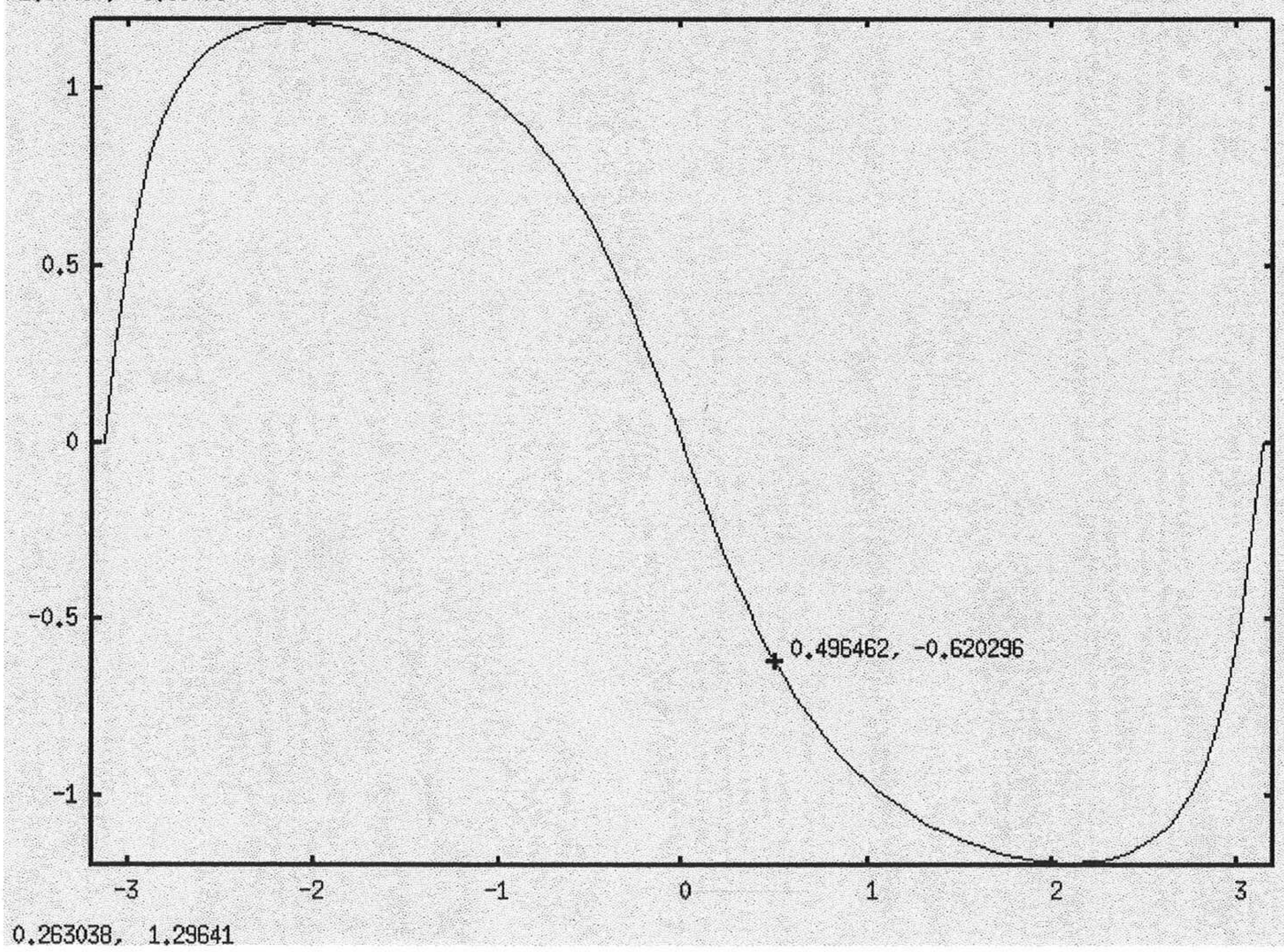
I could send the eigen signals through the system and get the eigenvalues $H(e^{\pm j0.5})$. I'll do it how the book does though, because it is simpler. Notice how this is just like phasor analysis. (That's because it is exactly phasor analysis. In phasor analysis, we send eigensignals into our system and compute eigenvalues. The "difference" is that the eigensignal is implicit, and we only work with the eigen values explicitly.)

$$\underbrace{1 \cdot e^{-j\pi/3}}_{\substack{\text{constant}\\(\text{phasor})}} \xrightarrow{\mathcal{H}} \underbrace{\left|H(e^{j0.5})\right| e^{j\arg(H(e^{j0.5}))}}_{\substack{\text{eigenvalue}\\(\text{phasor response})}} \underbrace{e^{-j\pi/3}}_{\text{constant}}$$

$$H(e^{j0.5}) = \frac{e^{j0.5} + 0.8}{e^{j0.5} - 0.5} = 2.3180 - j1.6735$$
$$= 2.8590\, e^{j(-0.62532)}$$

$$\boxed{y[n] = 2.859 \cos(0.5n - \pi/3 - 0.62532)}$$


3.5
3
2.5
2
1.5
1
0.5
-3
-2
-1
0
1
2
3
0.496462, 2.88132
1.34480, 3.66475


1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
0.496462, -0.620296
0.263038, 1.29641